

Proper-time hypersurface of non-relativistic matter flows: Galaxy bias in general relativity

Jaiyul Yoo^{1,2*}

¹*Center for Theoretical Astrophysics and Cosmology, Institute for Computational Science, University of Zürich and*

²*Physics Institute, University of Zürich, Winterthurerstrasse 190, CH-8057, Zürich, Switzerland*

We compute the second-order density fluctuation in the proper-time hypersurface of non-relativistic matter flows and relate it to the galaxy number density fluctuation, providing physical grounds for galaxy bias in the context of general relativity. At the linear order, the density fluctuation in the proper-time hypersurface is equivalent to the density fluctuation in the comoving synchronous gauge, in which two separate gauge conditions coincide. However, at the second order, the density fluctuations in these gauge conditions differ, while both gauge conditions represent the same proper-time hypersurface. Compared to the density fluctuation in the temporal comoving and the spatial C-gauge conditions, the density fluctuation in the commonly used gauge condition ($N = 1$ and $N^\alpha = 0$) violates the mass conservation at the second order. We provide their physical interpretations in each gauge condition by solving the geodesic equation and the nonlinear evolution equations of non-relativistic matter. We apply this finding to the second-order galaxy biasing in general relativity, which complements the second-order relativistic description of galaxy clustering in Yoo & Zaldarriaga (2014).

PACS numbers: 98.80.-k, 98.65.-r, 98.80.Jk, 98.62.Py

I. INTRODUCTION

The discovery of the late-time cosmic acceleration has created the problem of the century in physics, spurring numerous theoretical and observational investigations in the last decades. In particular, enormous amount of efforts have been devoted to large-scale galaxy surveys that can be used to map the three-dimensional matter distribution. Millions of galaxies at higher redshift with larger sky coverage will be measured in the upcoming future surveys such as the Dark Energy Spectroscopic Instrument, the Large Synoptic Survey Telescope, and two space-based missions Euclid and the Wide-Field Infrared Survey Telescope.

In light of this recent development in large-scale galaxy surveys, the general relativistic description of galaxy clustering has been developed [1–6]. In the standard Newtonian description, gravity is felt instantaneously across the horizon, and a hypersurface of simultaneity is well defined. However, none of these are valid in general relativity, and the Newtonian description breaks down on cosmological scales, in which dark energy models manifest themselves or modified gravity theories deviate from general relativity. The relativistic description is, therefore, an indispensable tool in the era of precision cosmology.

Cosmological observations are performed by measuring photons emitted from distant sources like galaxies, and they are affected by the matter fluctuation and the gravitational perturbations along the path to reach us. Of significant interest is, therefore, to derive the relation of the observable quantities to the physical quantities of sources and to understand how various relativistic effects such as gravitational potential and curvature perturbation affect this relation along the light propagation. In this way, the relativistic description of galaxy clustering naturally resolves gauge issues [1, 2] that often plague theoretical predictions, providing a complete de-

scription [7] of all the effects in galaxy clustering such as the redshift-space distortion, the gravitational lensing, the integrated Sachs-Wolfe effect, and so on (see [8] for a review).

A significant portion of the relativistic effects in galaxy clustering arise due to the mismatch between the physical quantities and the observable quantities, and this mismatch is tackled by tracing the light propagation backward in time and by deriving their relations. However, what we measure is galaxies, not matter, and the relation between the galaxy and the underlying matter distributions, known as *galaxy bias*, is another difficulty in formulating the relativistic description of galaxy clustering. In general, the physical galaxy number density can be separated into the mean $\bar{n}_g(\tau)$ and the fluctuation δ_g^{int} around it:

$$n_g = \bar{n}_g(\tau)(1 + \delta_g^{\text{int}}), \quad (1)$$

and the linear bias model [9] in Newtonian dynamics shows that the intrinsic galaxy fluctuation should be proportional to the matter density fluctuation δ_m on sufficiently large scales:

$$\delta_g^{\text{int}} = b \delta_m, \quad (2)$$

where the proportionality constant b is the bias factor. Since the separation of the mean and the fluctuation in Eq. (1) is arbitrary and relies on unspecified coordinate time τ , the biasing relation in Eq. (2) makes little sense in the context of general relativity and is gauge-dependent.

From the relativistic perspective, Yoo, Fitzpatrick, and Zaldarriaga [1] assumed that the galaxy number density is a function of the matter density ρ_m (not the matter density fluctuation δ_m) at the same spacetime point:

$$n_g = F[\rho_m]. \quad (3)$$

The galaxy biasing in Bonvin and Durrer [4] and Bruni et al. [10] is neglected or assumed to follow the matter density, respectively (hence it is essentially equivalent to Eq. [3] with F being the identity function). While this biasing scheme is fully general and covariant, it is physically restrictive as the

*jyoo@physik.uzh.ch

time evolution of galaxy number density is strictly driven by the matter density evolution $\bar{n}_g \propto (1+z)^3$. To relax this restriction, while keeping the locality, additional freedom was provided in Yoo et al. [6] to allow galaxy number density to depend on its local history (or *proper-time*), describing different evolutionary tracks of galaxy number densities at the same matter density.

By arguing that the galaxy number density is a Newtonian gauge quantity and its Poisson equation is related to the matter density fluctuation δ_m^{syn} in the synchronous gauge, Challinor and Lewis [3] chose the synchronous gauge for galaxy bias in general relativity, and the biasing relation in Eq. (2) becomes $\delta_g^{\text{int}} = b \delta_m^{\text{syn}}$. Jeong et al. [5] advocated the constant-age hypersurface (or the proper-time hypersurface) for the biasing relation, as the proper-time is the *only* locally measurable quantity that carries physical significance on large scales. A proper generalization in the context of general relativity is made in Baldauf et al. [11] by constructing a local Fermi coordinate, in which local observables can be explicitly written in terms of the local curvature and the local expansion rate. These biasing schemes based on the proper-time hypersurface [3, 5, 6, 11] are all equivalent to each other at the linear order.

Given these theoretical developments, it is rather straightforward, albeit lengthy, to extend the relativistic formalism to the second order in perturbation. The second-order perturbations are naturally smaller than the linear-order perturbations. However, they do contain critical and invaluable information about the perturbation generation mechanism in the early Universe. In the standard single-field inflationary model, the Universe is well described by its nearly perfect Gaussianity on large scales, in which the power spectrum contains the complete information. However, any models beyond the single-field inflationary model have additional degrees-of-freedom, and these additional fields couple to the curvature perturbations, leaving non-trivial signatures manifest in higher-order statistics such as the bispectrum (see, e.g., [12–14]). Even in the standard single field model, gravity waves generate non-trivial trispectrum in the curvature perturbations [15]. These unique signatures in the initial condition are generically subtle and nonlinear relativistic effects, requiring proper relativistic treatments beyond the linear-order in perturbations. In this respect, the second-order relativistic description of galaxy clustering provides an essential tool to probe the early Universe in large-scale galaxy surveys, and it was recently formulated [16–18].

Bertacca et al. [17] advocate that the proper-time hypersurface (or the rest-frame of baryons and dark matter) should be used for second-order galaxy bias, and they chose the matter density fluctuation δ_m^{II} in the comoving-time orthogonal gauge (see our gauge choice II in Table I) that becomes comoving-synchronous gauge for a pressureless medium. In Di Dio et al. [18], the galaxy number density is approximated as the matter density, and the second-order galaxy biasing is left for future work. In Yoo and Zaldarriaga [16], the proper-time hypersurface is also advocated for the second-order galaxy biasing scheme, but no specific choice of gauge condition is discussed for computing the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface at the second order.

Here we provide the missing ingredient, completing the full second-order relativistic description in [16]. We compute the second-order matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface of non-relativistic matter flows. In particular, we focus on several gauge choices summarized in Table I in computing $\delta_m^{t_p}$. Interestingly, these common gauge conditions provide different matter density fluctuations at the second order, posing a critical question in formulating galaxy bias in general relativity — *which one and why? any gauge issues?* We show that the matter density fluctuation in gauge choice I in Table I is the correct and physical choice for the matter density fluctuation in the proper-time hypersurface that can be used for galaxy bias in general relativity at the second order.

Technical details of these gauge conditions at the second order are extensively discussed in Hwang and Noh [19] (see also [20, 21] and [22] for different derivations). Here we provide physical interpretations of each gauge condition and discuss how they can be applied to second-order galaxy bias in general relativity. The organization of the paper is as follows. In Sec. II, we present the basic formalism for computing the flow of non-relativistic matter and derive the matter density fluctuation in the proper-time hypersurface by solving the geodesic equation. Various observers are defined in Sec. II C. Several gauge choices in Table I and the gauge issues associated with them are discussed in Sec. III in computing the matter density fluctuation in the proper-time hypersurface. In Sec. IV, we present the nonlinear evolution equations and derive their solutions for each gauge choice. Gauge issues in the solutions and their physical interpretation are discussed in Sec. V and Sec. VI, respectively. Finally, we summarize our finding and discuss the implications of our results in Sec. VIII. Two appendices summarize useful relations that are used in the paper.

Throughout the paper spacetime indices are represented by Latin indices, while spatial indices by Greek indices. Equations and variables in this paper should be considered nonlinear, unless perturbation order is specifically mentioned.

II. FLOW OF NON-RELATIVISTIC MATTER

Here we present the formalism for describing non-relativistic matter flows in cosmology and derive the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface.

A. Spacetime metric

We first define the spacetime metric g_{ab} , on which our calculations rely. The background universe is described by the usual FRW metric and small departures from the homogeneous and isotropic universe are captured by metric perturbations

$$\delta g_{00} = -2\mathcal{A}, \quad \delta g_{0\alpha} = -a\mathcal{B}_\alpha, \quad \delta g_{\alpha\beta} = 2a^2\mathcal{C}_{\alpha\beta}, \quad (4)$$

where the zeroth coordinate is the proper time t (not the conformal time), the scale factor is $a(t)$, and the perturbations \mathcal{B}_α

and $C_{\alpha\beta}$ are based on the three-metric $\bar{g}_{\alpha\beta}$ in the background. The departures in the metric are defined in a non-perturbative way, and hence each variable can be perturbatively split at each order, e.g.,

$$\mathcal{A} = \mathcal{A}^{(1)} + \mathcal{A}^{(2)} + \mathcal{A}^{(3)} + \dots \quad (5)$$

According to the generalized Helmholtz equation [23, 24], we further decompose the perturbation variables into scalar (β , φ , γ), transverse vector (B_α , C_α), and traceless transverse tensor ($C_{\alpha\beta}$) as

$$\mathcal{B}_\alpha = \beta_{,\alpha} + B_\alpha, \quad \mathcal{C}_{\alpha\beta} = \varphi \bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C_{(\alpha|\beta)} + C_{\alpha\beta}, \quad (6)$$

where the round bracket is the symmetrization and the comma and the vertical bar are the spatial derivative and the covariant derivative with respect to $\bar{g}_{\alpha\beta}$, respectively. It is noted that the decomposition is also independent of perturbation orders [16], and their spatial indices make the separation of scalar, vector, and tensor apparent.

B. ADM formalism

As we need to work on higher-order perturbations, it proves convenient to work with the Arnowitt-Deser-Misner (ADM) formalism [25, 26] and to derive fully nonlinear equations before we perform perturbative calculations. Its connection to the perturbed FRW metric is given in Appendix A.

In the ADM formalism, the spacetime is split into an ordered sequence of hypersurfaces labeled by a time coordinate t , and the intrinsic geometry of hypersurfaces is represented by the spatial metric $h_{\alpha\beta} = g_{\alpha\beta}$. The proper-time $\Delta\tau$ between two hypersurfaces separated by Δt is characterized by the lapse function N , and the shift vector N^α describes the spatial coordinate change of a normal direction between the hypersurfaces:

$$\Delta\tau = N\Delta t, \quad \Delta x^\alpha = N^\alpha \Delta t. \quad (7)$$

Therefore, the spacetime metric in the ADM formalism is described as

$$ds^2 = g_{ab} dx^a dx^b = -N^2 dt^2 + \gamma_{\alpha\beta} (dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt), \quad (8)$$

where the individual metric components are

$$g_{00} = -N^2 + N^\alpha N_\alpha, \quad g_{0\alpha} = N_\alpha = h_{\alpha\beta} N^\beta, \quad g_{\alpha\beta} = \gamma_{\alpha\beta}, \quad (9)$$

and their inverse components are

$$g^{00} = -\frac{1}{N^2}, \quad g^{0\alpha} = \frac{N^\alpha}{N^2}, \quad g^{\alpha\beta} = \gamma^{\alpha\beta} - \frac{N^\alpha N^\beta}{N^2}. \quad (10)$$

Given the 3+1 split in the ADM formalism, the local bending of spacelike hypersurfaces in spacetime is described by the extrinsic curvature

$$K_{\alpha\beta} = \frac{1}{2N} (N_{\alpha;\beta} + N_{\beta;\alpha} - \gamma_{\alpha\beta,0}) = -N\Gamma_{\alpha\beta}^0, \quad (11)$$

where Γ_{bc}^a is the Christoffel symbol based on g_{ab} and the colon is the covariant derivative with respect to $\gamma_{\alpha\beta}$. The extrinsic curvature can be further split into the trace part $K = \gamma^{\alpha\beta} K_{\alpha\beta}$ and the traceless part $\bar{K}_{\alpha\beta}$ as

$$\bar{K}_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{3} \gamma_{\alpha\beta} K. \quad (12)$$

C. Different observers

In cosmology, many different observers can be defined in describing the fluid and the metric quantities, although each observer may not be related to real observation. Here we clarify the difference by providing the exact definitions for later use, while keeping the terminology “observers.”

In the ADM formalism, the normal observer is defined by the flow of the normal direction of spatial hypersurfaces

$$n_a = (-N, 0), \quad n^a = \left(\frac{1}{N}, -\frac{1}{N} N^\alpha \right). \quad (13)$$

The normal observer is indeed the normal vector of hypersurfaces in 3+1 split, the flow of which is related to the extrinsic curvature

$$K_{\alpha\beta} = -n_{\alpha;\beta}, \quad (14)$$

where the semicolon denotes the covariant derivative with respect to g_{ab} in spacetime. The induced metric on the hypersurface is, therefore, $\gamma_{\alpha\beta} = g_{\alpha\beta} + n_\alpha n_\beta = g_{\alpha\beta}$. The normal observer, as defined in a given coordinate system, is a geometric quantity, but is not necessarily related to any flow of matter.

In general, a four velocity vector u^a can be defined to describe the flow of any fluids in cosmology

$$u^0 \equiv 1 + \delta u^0, \quad u^\alpha \equiv \frac{1}{a} V^\alpha, \quad (15)$$

where the perturbations $(\delta u^0, V^\alpha)$ are defined with respect to the case in a homogeneous universe (hence based on $\bar{g}_{\alpha\beta}$) and they are subject to the time-like normalization condition ($u^a u_a = -1$; similarly for the normal observer). If spatial velocity vector V^α is the velocity of a fluid component, the observer described by u^α moves together with the fluid and is called the comoving observer. Here we will consider the case in which the observer with u^a always moves together with the fluid, hence the comoving observer. However, two velocities can be different in principle, and the observer with u^a may not be necessarily comoving with any fluids.

Another observer of interest is the coordinate observer ($V^\alpha \equiv 0$), whose motion is fixated at a given spatial coordinate (hence the name). Same as for the normal observer, the coordinate observer is not directly related to any flow of matter. For later convenience, we define the covariant spatial component of the four velocity vector

$$u_\alpha = g_{\alpha b} u^b \equiv a(-v_{,\alpha} + v_\alpha), \quad (16)$$

in terms of scalar v and vector v_α components. The four velocity vector u^a can be used to describe the normal vector n^a , if $v = v_\alpha = 0$, which is called the comoving gauge condition (see Sec. III).

It is evident from the definition of various observers that the comoving observer is physically relevant to the evolution of fluids in cosmology, while the normal and the coordinate observers describe the geometry of a given spacetime metric and coordinate system.

D. Covariant decomposition and energy-momentum tensor

Any four velocity vector can be covariantly decomposed into physically well-defined quantities of flows described by u^a [27, 28]

$$u_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab} + \omega_{ab} - a_a u_b, \quad (17)$$

where the expansion and the acceleration of the flow are $\theta = u^a{}_{;a}$, $a_a = u^b u^a{}_{;b}$, the projection tensor $h_{ab} = g_{ab} + u_a u_b$, and the shear and the rotation of the flow are $\sigma_{ab} = u_{(a;b)} + a_{(a} u_{b)}$, and $\omega_{ab} = u_{[a;b]} + a_{[a} u_{b]}$. A similar decomposition is possible for the normal observer n^a , and these covariant quantities represent the geometry of the hypersurface in spacetime:

$$\theta = -K, \quad \omega_{ab} = 0, \quad \sigma_{\alpha\beta} = -\bar{K}_{\alpha\beta}. \quad (18)$$

The energy-momentum tensor of fluids can be written in full generality [27, 28]

$$\mathcal{T}_{ab} = \rho u_a u_b + p h_{ab} + 2q_{(a} u_{b)} + \pi_{ab}, \quad (19)$$

where ρ and p are the energy density and isotropic pressure of the fluid, q_a is the energy flux, and π_{ab} is the anisotropic pressure. Those fluid quantities are ones measured by the observer described by u^a :

$$\begin{aligned} \rho &= \mathcal{T}_{ab} u^a u^b, & p &= \frac{1}{3} \mathcal{T}_{ab} h^{ab}, \\ q_a &= -\mathcal{T}_{cd} u^c h_a^d, & \pi_{ab} &= \mathcal{T}_{cd} h_a^c h_b^d - p h_{ab}, \end{aligned} \quad (20)$$

and hence they are frame-dependent (or observer dependent) [20]. Therefore, it is most convenient to use the fluid quantities measured by the comoving observer, or the fluid quantities in the rest frame, i.e., $q_a = 0$.

Here we will focus on the pressureless medium of non-relativistic matter with $p = \pi_{ab} = 0$, a good approximation to the late-time Universe on large scales, where baryons are effectively pressureless. Hence the energy-momentum tensor simplifies as

$$\mathcal{T}_{ab} = \rho_m u_a u_b. \quad (21)$$

Furthermore, we will consider an irrotational fluid $\omega_{ab} = 0$, which dictates that the vector component of the four velocity should vanish

$$v_\alpha = 0. \quad (22)$$

E. Geodesic motion of non-relativistic matter

We are interested in the motion of non-relativistic matter described by the energy-momentum tensor in Eq. (21). Without pressure, the non-relativistic matter responds to the gravity only, following the geodesic path, and the geodesic equation is $a_\alpha = 0$:

$$\begin{aligned} a_\alpha &= \dot{u}_\alpha u^0 + u_{\alpha,\beta} u^\beta - \frac{u_0 u^0}{N} [N_{,\alpha} - K_{\alpha\beta} N^\beta] \\ &\quad - u_\beta u^0 \left[-\frac{1}{N} N_{,\alpha} N^\beta - N K_\alpha^\beta + N^\beta{}_{;\alpha} + \frac{1}{N} N^\beta N^\delta K_{\alpha\delta} \right] \\ &\quad + \frac{u_0 u^\beta}{N} K_{\alpha\beta} - u_\delta u^\beta \left[\Gamma_{\alpha\beta}^{(\gamma)\delta} + \frac{1}{N} N^\delta K_{\alpha\beta} \right], \end{aligned} \quad (23)$$

where $\Gamma_{\alpha\beta}^{(\gamma)\delta}$ is the Christoffel symbol based on $\gamma_{\alpha\beta}$. For the normal observer $u_\alpha = n_\alpha = 0$, the geodesic equation greatly simplifies as

$$a_\alpha = \frac{1}{N} N_{,\alpha} = 0. \quad (24)$$

In Sec. II C we considered different observers with four velocity vector u^a . For non-relativistic matter flows, the path described by u^a is timelike, and the normalization condition ($-1 = u^a u_a$) implies that the path can be parametrized by the affine parameter λ in proportion to the proper time τ , i.e., $u^a = dx^a/d\lambda$. Therefore, the path of the observers in spacetime can be obtained by integrating their velocity vector over the affine parameter λ

$$x_\lambda^a - x_{\lambda_o}^a = (t_\lambda - t_{\lambda_o}, x_\lambda^\alpha) = \int_{\lambda_o}^\lambda d\lambda' u^a, \quad (25)$$

where we set the spatial coordinate $x_{\lambda_o}^\alpha = 0$ at λ_o . To the zeroth order in perturbation, the spatial position remains unchanged $\delta x^\alpha = 0$, and the proper time elapsed along the fluid is related to the affine parameter

$$\Delta\tau = \bar{t}_\lambda - \bar{t}_{\lambda_o} = \lambda - \lambda_o. \quad (26)$$

In the presence of perturbations, the path of the observers drifts away from the background relation

$$x_\lambda^a - x_{\lambda_o}^a \equiv (\Delta\tau + \delta\tau, \delta x^\alpha), \quad (27)$$

and from Eq. (25) the spacetime drifts $\Delta x^a = (\delta\tau, \delta x^\alpha)$ are derived to the second order in perturbations as

$$\begin{aligned} \delta\tau &= \int_{\bar{t}_{\lambda_o}}^{\bar{t}} d\bar{t} \left[\delta u^0 + \Delta x^a \delta u^0{}_{,a} \right], \\ \delta x^\alpha &= \int_{\bar{\eta}_{\lambda_o}}^{\bar{\eta}} d\bar{\eta} \left[V^\alpha + a \Delta x^b \left(\frac{V^\alpha}{a} \right)_{,b} \right], \end{aligned} \quad (28)$$

where η is the conformal time and the overbar is used to indicate that the integration is along the background path. Note that the spacetime drifts Δx^a in the integrand should be evaluated at λ , not at the background. However, to the second order in perturbation, it can be evaluated at the background. While we focus on the geodesic motion of non-relativistic matter, the spacetime drifts in Eq. (28) are valid for flows with non-vanishing acceleration, as long as their path is timelike.

TABLE I: Gauge conditions considered in this paper

gauge choice	temporal gauge condition	spatial gauge condition	ADM variables	comoving observer	remaining gauge mode
I	comoving $v = 0$	$\gamma = C_\alpha = 0$ (C-gauge)	$N = 1, N_\alpha \neq 0$	normal	No
II	comoving $v = 0$	$\beta = B_\alpha = 0$ (B-gauge)	$N = 1, N_\alpha = 0$	normal, coordinate	Yes
III	synchronous $\mathcal{A} = 0$	$\beta = B_\alpha = 0$ (B-gauge)	$N = 1, N_\alpha = 0$.	Yes

F. Matter fluctuation in proper-time hypersurface

Having related the coordinate time t of observers to their locally measured proper-time τ , we can construct a hypersurface of same proper-time of non-relativistic matter and compute the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface. Since the matter density at a given spacetime point can be split into the background and the fluctuation around it,

$$\rho_m(x^a) = \bar{\rho}_m(t) [1 + \delta_m(x^a)] = \bar{\rho}_m(\tau) [1 + \delta_m^{t_p}] , \quad (29)$$

we derive $\delta_m^{t_p}$ to the second order in perturbation:

$$\delta_m^{t_p} = \delta_m - 3H\delta\tau(1 + \delta_m) + \frac{3}{2}(3H^2 - \dot{H})\delta\tau^2 , \quad (30)$$

where H is the Hubble parameter. It is noted that the expression is gauge-invariant at the linear order, as the time-slicing is fully specified. A proper-time hypersurface of non-relativistic matter flows is physically well-defined, corresponding to a complete choice of gauge condition. However, at the second order, the spatial gauge transformation affects perturbations, and the spacing of the hypersurface needs to be fully specified.

III. GAUGE CHOICE

Here we describe gauge choices one can make in computing the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface. Since the proper-time hypersurface of non-relativistic matter is physically well-defined, any gauge choice can be made to compute $\delta_m^{t_p}$ in Eq. (30). However, it would be preferable to make a gauge choice, in which the coordinate time represents the proper time of non-relativistic matter flows, i.e., $\delta\tau = 0$. Meanwhile, unphysical gauge modes may remain in the solutions for certain choices of gauge conditions. For instance, it is well-known that the synchronous gauge fails to completely fix gauge freedom and has gauge modes to the linear order (e.g., [29]). Here we consider three popular choices of gauge conditions summarized in Table I and discuss the geodesic motion of the comoving observer and the matter density fluctuation in each gauge choice. The second-order matter density fluctuations are derived in Sec. IV, and their gauge issues and physical interpretation are discussed in Sec. V and Sec. VI, respectively.

A. Gauge transformation

The principle of general covariance dictates that any coordinate system can be used to describe physics in general relativity. However, since the background quantities in cosmology depend only on the time coordinate due to symmetry, a change in coordinate systems accompanies a change in the correspondence to the background, and perturbations in a given coordinate system accordingly change [23].

Given a coordinate transformation,¹

$$\tilde{\eta} = \eta + T , \quad \tilde{x}^\alpha = x^\alpha + \mathcal{L}^\alpha , \quad (32)$$

the scalar and the vector perturbations gauge transform to the linear order as

$$\begin{aligned} \tilde{\mathcal{A}} &= \mathcal{A} - T' - \mathcal{H}T , & \tilde{\beta} &= \beta - T + L' , & \tilde{\varphi} &= \varphi - \mathcal{H}T , \\ \tilde{\gamma} &= \gamma - L , & \tilde{v} &= v - T , & \tilde{\delta}_m &= \delta_m + 3\mathcal{H}T , \\ \tilde{\chi} &= \chi - aT , & \tilde{\kappa} &= \kappa + \left(3\dot{H} + \frac{\Delta}{a^2} \right) aT , \\ \tilde{B}_\alpha &= B_\alpha + L'_\alpha , & \tilde{C}_\alpha &= C_\alpha - L_\alpha , \end{aligned} \quad (33)$$

where the prime is the derivative with respect to the conformal time, the conformal Hubble parameter is $\mathcal{H} = a'/a = aH$, and we further decomposed the spatial transformation into scalar L and vector L^α as

$$\mathcal{L}^\alpha = L^{\cdot\alpha} + L^\alpha . \quad (34)$$

For later reference, we defined $\chi = a(\beta + \gamma')$ and $\kappa = \delta K = 3(H\mathcal{A} - \dot{\varphi}) - \Delta\chi/a^2$. The spatial vector v_α and the tensor $C_{\alpha\beta}$ perturbations are gauge-invariant at the linear order. It is noted that the gauge-transformation relations in Eq. (33) are valid only to the linear order, and we will consider second-order gauge-transformation in Sec. V.

Gauge freedoms expressed in terms of T and \mathcal{L}^α need to be fully removed by an appropriate choice of gauge conditions. Otherwise, perturbation variables are not uniquely defined, as illustrated in Eq. (33). We use *temporal* and *spatial* gauge conditions to refer to the gauge conditions fixing the temporal T and the spatial \mathcal{L}^α gauge freedoms, respectively.

¹ Here we use the conformal time η in considering a coordinate transformation, instead of the proper time t . The relation for two different coordinate transformations can be readily derived as

$$T_t = aT_\eta + \frac{1}{2}a'T_\eta^2 + \dots , \quad (31)$$

where T_t and T_η are defined in relation to their coordinate transformations.

B. Gauge choice I: Temporal comoving and spatial C-gauge

We consider the first gauge choice in Table I, in which the temporal gauge condition is set by $v = 0$ of non-relativistic matter flows and the spatial gauge condition is set by $\gamma = C_\alpha = 0$ (C-gauge). With the irrotational condition of the fluid, the temporal gauge condition $v = 0$ implies the covariant spatial component of the observer vanishes $u_\alpha = 0$ in Eq. (16) and the energy-momentum tensor in Eq. (21) is

$$\mathcal{T}_{ab} = N^2 \rho_m \delta_a^0 \delta_b^0, \quad \mathcal{T}_\alpha^0 = 0. \quad (35)$$

Therefore, this gauge choice is often called the *comoving gauge*, as the rest-frame comoving observer sees a vanishing energy flux $\mathcal{T}_\alpha^0 = 0$. Furthermore, the four velocity vector of the comoving observer in this case describes the normal observer $u^a = n^a$, which differs from the coordinate observer, though.

It is apparent in Eq. (33) that the temporal comoving gauge condition sets $T = 0$ and the spatial C-gauge condition sets $L^\alpha = L = 0$, completely eliminating the gauge freedom to the linear order. To the second order in perturbation, we will choose $v = 0$ as our temporal gauge condition. It was explicitly shown [16, 20] (see also Eqs. [70] and [71] in Sec. V) that the spatial gauge condition $\gamma = C_\alpha = 0$ to the higher-order in perturbation completely fixes the gauge freedom $T = L = L^\alpha = 0$ if the linear-order gauge condition that sets $T = 0$ is chosen to the higher-order in perturbation.

From the geodesic condition in Eq. (24), the lapse function can be set $N = N(t) = 1$ (see Sec. V and Appendix A). The metric perturbations and the ADM variables in this gauge choice are

$$\mathcal{B}_\alpha = \frac{1}{a} \chi_{,\alpha} + \Psi_\alpha, \quad \mathcal{C}_{\alpha\beta} = \varphi \bar{g}_{\alpha\beta} + C_{\alpha\beta}, \quad (36)$$

$$N_\alpha = -\chi_{,\alpha} - a\Psi_\alpha, \quad h_{\alpha\beta} = a^2 [(1 + 2\varphi)\bar{g}_{\alpha\beta} + 2C_{\alpha\beta}],$$

where we defined the vector perturbation $\Psi_\alpha = B_\alpha + C'_\alpha$. To the linear order in perturbation Ψ_α is gauge-invariant and χ is spatially gauge-invariant, according to Eq. (33).

In this gauge choice, the comoving normal observer is

$$u^a = n^a = (1, -N^\alpha), \quad \delta u^0 = 0, \quad (37)$$

and the geodesic path of the observer is

$$x_\lambda^a - x_{\lambda_o}^a = (\Delta\tau, \delta x^\alpha), \quad \delta\tau = 0, \quad (38)$$

where the time drift vanishes and the spatial drift is

$$\delta x^\alpha = - \int_{\bar{t}_{\lambda_o}}^{\bar{t}} d\bar{t} \left[N^\alpha + \delta x^\beta N_{,\beta}^\alpha \right]. \quad (39)$$

Over some proper time $\Delta\tau$ measured by the comoving observer in the rest frame of non-relativistic matter, they drift away from the initial spatial position, but the time coordinate of the observer in this gauge condition is *synchronized* with the proper time. The matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface is, therefore,

$$\delta_m^{t_p} = \delta_m^I \quad \text{for gauge choice I.} \quad (40)$$

To the linear order in perturbation, the coordinate observer is identical to the comoving normal observer, and the same conclusion for $\delta_m^{t_p}$ can be drawn. However, at higher order, δu^0 of the coordinate observer is non-vanishing (hence $\delta\tau \neq 0$), and the coordinate observer has to accelerate ($a_\alpha \neq 0$) to stay at the same spatial coordinates. Despite the non-geodesic motion of gauge choice I, as described by the coordinate observer, the matter density fluctuation in gauge choice I represents that of the proper-time hypersurface, because it is the comoving observer of non-relativistic matter that is physically relevant and whose proper time is synchronized with coordinate time.

C. Gauge choice II: Temporal comoving and spatial B-gauge

The second gauge choice in Table I is a variant of the comoving gauge, which is identical in the temporal gauge condition $v = 0$ but differs only in the spatial gauge condition $\mathcal{B}_\alpha = 0$ ($\beta = B_\alpha = 0$; B-gauge). Therefore, the metric perturbations and the ADM variables in this gauge choice are

$$\begin{aligned} \mathcal{B}_\alpha &= N_\alpha = 0, \quad h_{\alpha\beta} = a^2 (\bar{g}_{\alpha\beta} + 2C_{\alpha\beta}), \quad (41) \\ \mathcal{C}_{\alpha\beta} &= \varphi \bar{g}_{\alpha\beta} + \gamma_{,\alpha|\beta} + C_{(\alpha|\beta)} + C_{\alpha\beta}, \end{aligned}$$

where no further simplification is possible for $\mathcal{C}_{\alpha\beta}$.

With the same temporal gauge condition, gauge choice II is again the comoving gauge $\mathcal{T}_\alpha^0 = 0$, and the comoving observer coincides with the normal observer. With vanishing shift function $N_\alpha = 0$, it is also the coordinate observer ($V^\alpha = 0$) in this case. The geodesic condition in Eq. (24) implies that $N = 1$ and $\mathcal{A} = 0$ (see Sec. V), and hence gauge choice II is often called the *comoving-synchronous gauge*.

However, as is apparent in Eq. (33), while the temporal gauge freedom is removed $T = 0$, the spatial gauge freedom in this case is constrained only to its derivative, i.e., $\mathcal{L}'_\alpha = 0$, implying that even to the linear order in perturbation there remain spatial gauge modes $L = L(\mathbf{x})$ and $L_\alpha = L_\alpha(\mathbf{x})$, such that γ and C_α are uniquely determined up to any time-independent, but *scale-dependent* functions. We discuss how the remaining gauge modes affect the solutions in Sec. V.

The motion of the comoving observer is simpler in this gauge choice — the observer four velocity vector and its path are

$$u^a = n^a = (1, 0), \quad x_\lambda^a - x_{\lambda_o}^a = (\Delta\tau, 0), \quad (42)$$

and with vanishing time drift $\delta\tau = 0$ the matter density fluctuation δ_m in this gauge choice again represents the matter density fluctuation $\delta_m^{t_p}$ in the same proper-time hypersurface

$$\delta_m^{t_p} = \delta_m^{II} \quad \text{for gauge choice II.} \quad (43)$$

For this simplicity, gauge choice II has been widely used in literature for computing nonlinear equations (e.g., [30]). We show in Sec. V that the remaining gauge modes in gauge choice II affect the matter density fluctuation, and hence it is incomplete. While we can project out the gauge mode in δ_m^{II} , we show that the matter density fluctuations in gauge choices I and II are different in Sec. IV and its physical interpretation is presented in Sec. VI.

D. Gauge choice III: The synchronous gauge

The third gauge choice in Table I is the original *synchronous gauge*, in which the temporal gauge condition is set by $\mathcal{A} = 0$ and the spatial gauge condition is $\mathcal{B}_\alpha = 0$ ($\beta = B_\alpha = 0$; B-gauge). This gauge condition implies that the ADM variables are $N = 1$ and $N_\alpha = 0$. Similarly in gauge choice II, the normal observer is the coordinate observer. However, the comoving observer in this case is

$$u^a = \left(1 + \frac{1}{2}V^\alpha V_\alpha, \frac{1}{a}V^\alpha\right), \quad (44)$$

different from the normal observer. It is apparent that the geodesic motion of the comoving observer is not synchronous with the coordinate time ($\delta\tau \neq 0$) as $\delta u^0 \neq 0$, and the matter density fluctuation δ_m in this gauge choice differs from $\delta_m^{t_p}$ of the proper-time hypersurface, unless the spatial velocity vector vanishes $V^\alpha = 0$. Nevertheless, Eq. (30) can be used to compute $\delta_m^{t_p}$ in gauge choice III, despite $\delta\tau \neq 0$ (hence $\delta_m^{t_p} \neq \delta_m$).

Moreover, it is well-known that the synchronous gauge fails to fix the gauge freedom. To the linear order in perturbation, the gauge transformation in Eq. (33) only constrains the gauge freedom as

$$T = \frac{c_1(\mathbf{x})}{a}, \quad L = c_1(\mathbf{x}) \int^t \frac{dt}{a^2} + c_2(\mathbf{x}), \quad L_\alpha = L_\alpha(\mathbf{x}), \quad (45)$$

where $c_i(\mathbf{x})$ is a time-independent but scale-dependent function. Therefore, the all the perturbation variables other than $\mathcal{A} = \mathcal{B}_\alpha = 0$ have remaining unphysical gauge modes, even at the linear order.

However, the geodesic condition in Eq. (23) yields that the spatial velocity vector decays in time

$$V^\alpha \propto \frac{1}{a}, \quad (46)$$

which suggests that by imposing the initial condition $V^\alpha = 0$ at some early time (e.g., see [29, 31]), the spatial velocity vanishes all the time, and the comoving observer in Eq. (44) becomes the coordinate observer (and the normal observer). Indeed, this initial condition makes gauge choice III identical to gauge choice II, as the covariant spatial component also vanishes $v = 0$ (see Appendix B). Unless the comoving gauge condition $v = 0$ is imposed, gauge choice III is complicated and plagued with unphysical gauge modes beyond the linear order in perturbation. Hereafter, we assume the specific initial condition is adopted for gauge choice III (hence identical to gauge choice II), and further discussion of gauge choice III will be referred to gauge choice II.

IV. NONLINEAR EVOLUTION EQUATIONS OF THE MATTER DENSITY FLUCTUATION

Here we derive the nonlinear equation, governing the irrotational pressureless fluid [19, 22, 32, 33] and obtain their solutions for the gauge choices in Table I. In both gauge choices I

and II, the matter density fluctuations δ_m represent the matter density fluctuation $\delta_m^{t_p}$ of the proper-time hypersurface. However, we show that the second-order solutions δ_m in those gauge choices are different from each other.

Using the covariant decomposition in Eq. (17), the conservation of the energy-momentum tensor in Eq. (21) yields that the irrotational pressureless fluid should follow the geodesic path and the energy density is conserved along the geodesic motion:

$$a_a = 0, \quad \frac{d}{d\lambda}\rho + \rho\theta = 0, \quad (47)$$

where the derivative with respect to the affine parameter is $d/d\lambda = u^b \partial_b$. It is noted that the geodesic condition used in Sec. III is the consequence of the energy-momentum conservation. The evolution of the expansion θ along the flow is described by the Raychaudhuri equation [34], and it simplifies for the irrotational pressureless medium as

$$\frac{d}{d\lambda}\theta + \frac{1}{3}\theta^2 + \sigma_{ab}\sigma^{ab} + R_{ab}u^a u^b = 0, \quad (48)$$

where the Ricci tensor R_{ab} can be further related to the energy-momentum tensor \mathcal{T}_{ab} by using the Einstein equation

$$R_{ab}u^a u^b = 4\pi G\rho_m - \Lambda. \quad (49)$$

These nonlinear equations are sufficient to describe the evolution of the irrotational pressureless fluid, and they can be readily solved by splitting into the background and the perturbation. The background equations are

$$\begin{aligned} 0 &= \dot{\bar{\rho}}_m + 3H\bar{\rho}_m, \\ 0 &= 3(\dot{H} + H^2) + 4\pi G\bar{\rho}_m - \Lambda, \end{aligned} \quad (50)$$

and the nonlinear perturbation equations are

$$\begin{aligned} \dot{\delta}_m - \kappa &= N^\alpha \delta_{m,\alpha} + \delta_m \kappa, \\ \dot{\kappa} + 2H\kappa - 4\pi G\bar{\rho}_m \delta_m &= N^\alpha \kappa_{,\alpha} + \frac{1}{3}\kappa^2 + \sigma^{ab}\sigma_{ab}, \end{aligned} \quad (51)$$

where the expansion of the normal observer is related to the perturbation $\kappa = \delta K$ of the extrinsic curvature K as

$$\theta = -K = 3H - \kappa. \quad (52)$$

Combining the two equations, the differential equation for the evolution of non-relativistic matter can be derived as

$$\begin{aligned} \ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m \delta_m \\ = \frac{1}{a^2} \left[a^2 (N^\alpha \delta_{m,\alpha} + \delta_m \kappa) \right]' + N^\alpha \kappa_{,\alpha} + \frac{1}{3}\kappa^2 + \sigma^{ab}\sigma_{ab}. \end{aligned} \quad (53)$$

These equations are derived by assuming the temporal comoving gauge ($v = 0$) with the normal observer ($u^a = n^a$), while the spatial gauge condition is left unspecified. Therefore, they apply to both gauge choices I and II in Table I.

A. Gauge choice I: Temporal comoving and spatial C-gauge

In gauge choice I, the shift function N^α is given in Sec. III B, and the source terms in Eq. (51) are computed in

$$\begin{aligned}\dot{\delta}_m - \kappa &= -\frac{1}{a^2}\chi^{\cdot\alpha}\delta_{m,\alpha} - \frac{1}{a}\Psi^\alpha\delta_{m,\alpha} + \delta_m\kappa, \\ \dot{\kappa} + 2H\kappa - 4\pi G\bar{\rho}_m\delta_m &= -\frac{1}{a^2}\chi^{\cdot\alpha}\kappa_{,\alpha} - \frac{1}{a}\Psi^\alpha\kappa_{,\alpha} + \left(\frac{1}{a^2}\chi_{,\alpha|\beta} + \frac{1}{a}\Psi_{\alpha|\beta} + \dot{C}_{\alpha\beta}\right)\left(\frac{1}{a^2}\chi^{\cdot\alpha|\beta} + \frac{1}{a}\Psi^{\alpha|\beta} + \dot{C}^{\alpha\beta}\right),\end{aligned}\tag{54}$$

where we used $\dot{\varphi} = 0$ at the linear order in computing the quadratic source terms. These coupled evolution equations are sourced not only by the scalar contributions, but also by the vector Ψ_α and the tensor $C_{\alpha\beta}$ contributions. While the evolution equations for the vector and tensor contributions can be supplemented, we simplify the nonlinear evolution equations (54) by neglecting the *linear-order* vector and tensor contributions, as we are interested in the evolution of non-relativistic matter in the late time. Furthermore, since the expansion perturbation is $\kappa = -\nabla^2\chi/a^2$ at the linear order (see Appendix A), the nonlinear evolution equations become a closed system of two differential equations for δ_m and κ .

To the second order in perturbation, we define a velocity vector \mathbf{v} in relation to the expansion perturbation as

$$\kappa \equiv -\frac{1}{a}\nabla \cdot \mathbf{v}, \tag{55}$$

and the velocity vector is curl-free to the linear order:

$$\mathbf{v}^{(1)} = \frac{1}{a}\nabla\chi^{(1)}. \tag{56}$$

In terms of the velocity vector, the nonlinear evolution equations for the matter density fluctuation and the expansion perturbation in gauge choice I become identical to the Newtonian equations for a pressureless medium [35, 36]:

$$\dot{\delta}_m + \frac{1}{a}\nabla \cdot \mathbf{v} = -\frac{1}{a}\nabla \cdot (\delta_m \mathbf{v}), \tag{57}$$

$$\nabla \cdot \dot{\mathbf{v}} + H\nabla \cdot \mathbf{v} + 4\pi G a \bar{\rho}_m \delta_m = -\frac{1}{a}\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}],$$

if we identify the matter density fluctuation δ_m and the expansion perturbation κ in gauge choice I as the Newtonian matter density fluctuation and the Newtonian expansion $\Theta = \nabla \cdot \mathbf{v}$ as in Eq. (55). With this identification, the master equation for the matter density fluctuation in Eq. (53) can be rephrased as

$$\begin{aligned}\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m \\ = -\frac{1}{a^2}[a\nabla \cdot (\delta_m \mathbf{v})] + \frac{1}{a^2}\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}].\end{aligned}\tag{58}$$

The correspondence between the Newtonian dynamics and the relativistic dynamics in gauge choice I is valid only to the second order in perturbation in the absence of *linear-order*

Appendix A. Therefore, the nonlinear evolution equations for the matter density fluctuation δ_m and the expansion perturbation κ can be written in terms of metric to the second order in perturbation as

vector and tensor components in a universe with irrotational pressureless medium, while it is noted that the Newtonian equations as in Eq. (57) are fully nonlinear (see, e.g., [37]), valid to all orders in perturbation. Beyond the second order, however, no exact correspondence is possible due to relativistic corrections (see, e.g., [36]). Second-order vectors and tensors are naturally generated by scalar contributions, and they affect the third-order scalar contributions. However, in the absence of the linear-order vector or tensor components, scalar-generated second-order vector or tensor components are decoupled from the nonlinear evolution equations (54).

B. Gauge choice II: Temporal comoving and spatial B-gauge

In gauge choice II, the metric perturbations are present only in the spatial metric $h_{\alpha\beta}$, and the nonlinear evolution equations (51) and (53) are simpler due to the absence of the shift function N^α . In terms of metric perturbations, the first two source terms in the right-hand side of Eqs. (54) are absent in gauge choice II, but its generic structure of the coupled differential equations remains unchanged — the evolution equations are closed, only if linear-order vector and tensor contributions vanish.

Under the same assumption that *no* vector and tensor contributions are present at the linear order, we express the evolution equations in terms of the Newtonian velocity vector defined in Eq. (55) as

$$\dot{\delta}_m + \frac{1}{a}\nabla \cdot \mathbf{v} = -\frac{1}{a}\delta_m \nabla \cdot \mathbf{v}, \tag{59}$$

$$\begin{aligned}\nabla \cdot \dot{\mathbf{v}} + H\nabla \cdot \mathbf{v} + 4\pi G a \bar{\rho}_m \delta_m &= -\frac{1}{a}\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] \\ &+ \frac{1}{a}\mathbf{v} \cdot \nabla(\nabla \cdot \mathbf{v}),\end{aligned}$$

and the master equation for the matter density fluctuation in Eq. (53) becomes

$$\begin{aligned}\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m \\ = -\frac{1}{a^2}[a\delta_m \nabla \cdot \mathbf{v}] + \frac{1}{a^2}\nabla \cdot [(\mathbf{v} \cdot \nabla)\mathbf{v}] - \frac{1}{a^2}\mathbf{v} \cdot \nabla(\nabla \cdot \mathbf{v}).\end{aligned}\tag{60}$$

As in gauge choice I, there exists a Newtonian correspondence in the evolution equations in gauge choice II [19].

As we showed in Sec. III, the spatial coordinates of non-relativistic matter remain unchanged all the time (the spatial drift $\delta x^\alpha = 0$), such that the coordinate system is tracking the particle motion, i.e., if we identify the time derivative in Eq. (59) as the Lagrangian derivative in Newtonian dynamics

$$\frac{\partial}{\partial t} \rightarrow \frac{D}{Dt} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla), \quad (61)$$

the evolution equations can be recasted as those in the Newtonian Lagrangian frame [19, 38, 39], describing the same system as in gauge choice I. However, this correspondence is again valid only to the second order in perturbation, in the absence of linear-order vector or tensor contributions. At the third order in perturbation, pure relativistic corrections appear [38].

C. Nonlinear solutions for matter density and expansion perturbation

Having set up the closed coupled differential equations, we derive the solutions for the matter density fluctuation and the expansion perturbation with the gauge choices in Table I. As the evolution equations are rephrased in a way similar to the Newtonian dynamics, we can simply follow the standard perturbative approach to solving the differential equations (e.g., [37]).

The matter density fluctuation and the divergence of the velocity vector (or the expansion) are expanded in Fourier space as

$$\begin{aligned} \delta_m(\mathbf{k}, t) &= \delta_m^{(1)} + \delta_m^{(2)} + \dots, \\ \Theta(\mathbf{k}, t) &= [\nabla \cdot \mathbf{v}](\mathbf{k}, t) = \Theta^{(1)} + \Theta^{(2)} + \dots, \end{aligned} \quad (62)$$

and the linear-order solutions takes the usual form

$$\delta_m^{(1)}(\mathbf{k}, t) = D(t)\delta(\mathbf{k}), \quad \Theta^{(1)}(\mathbf{k}, t) = -\mathcal{H}D(t)f\delta(\mathbf{k}), \quad (63)$$

where $\delta(\mathbf{k})$ is the matter density fluctuation at the initial time, the growth factor $D(t)$ is normalized at the initial time, and $f = d \ln D / d \ln a$ is the logarithmic growth rate.² The master equations (58) and (60) for the matter density fluctuation in gauge choices I and II are identical to the linear order, accommodating the same linear-order solution, as in Eq. (63).

The second-order solution can be derived by using the standard convolution forms in Fourier space as

$$\begin{aligned} \delta_m^{(2)}(\mathbf{k}, t) &= D^2(t) \int \frac{d^3 \mathbf{q}}{(2\pi)^3} F_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) \delta(\mathbf{q}) \delta(\mathbf{k} - \mathbf{q}), \\ \Theta^{(2)}(\mathbf{k}, t) &= -\mathcal{H}fD^2(t) \int \frac{d^3 \mathbf{q}}{(2\pi)^3} G_2(\mathbf{q}, \mathbf{k} - \mathbf{q}) \delta(\mathbf{q}) \delta(\mathbf{k} - \mathbf{q}). \end{aligned} \quad (64)$$

For gauge choice I, the perturbation kernels for the matter density fluctuation and the velocity divergence are

$$\begin{aligned} F_2^I(\mathbf{q}_1, \mathbf{q}_2) &= \frac{5}{7} + \frac{2}{7} \left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \right)^2 + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{2q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right), \\ G_2^I(\mathbf{q}_1, \mathbf{q}_2) &= \frac{3}{7} + \frac{4}{7} \left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \right)^2 + \frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{2q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right). \end{aligned} \quad (65)$$

Since the governing equations (57) coincide with the Newtonian dynamics, the perturbation kernels are naturally identical to those in the Newtonian perturbation theory. For gauge choice II, as the source terms in Eq. (59) are different from the standard Eulerian perturbation theory, the perturbation kernels are also different (see [38, 40, 41])

$$\begin{aligned} F_2^{II}(\mathbf{q}_1, \mathbf{q}_2) &= \frac{5}{7} + \frac{2}{7} \left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \right)^2, \\ G_2^{II}(\mathbf{q}_1, \mathbf{q}_2) &= \frac{3}{7} + \frac{4}{7} \left(\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{q_1 q_2} \right)^2. \end{aligned} \quad (66)$$

Compared to the perturbation kernels in Eqs. (65) for gauge choice I, the dipole terms are absent in Eqs. (66) for gauge choice II. The matter density fluctuations in the gauge choices in Table I are, therefore, different at the second-order. We discuss the difference in their physical interpretation in Sec. VI.

As is the case in the standard perturbation theory [37, 42] the recurrence relation can be derived for the perturbation kernels (F_n, G_n) at higher order in gauge choice I (e.g., [40, 43]). However, in the relativistic dynamics, the nonlinear equations (51) are closed to the second order only when the linear-order vector and tensor contributions are neglected. Furthermore, the shear amplitude $\sigma^{ab}\sigma_{ab}$ in the source term has additional contributions from δ_m and \mathbf{v} at orders beyond the second order [36], invalidating the use of the recurrence relation at orders higher than that of the shear amplitude. More importantly, beyond the second order in perturbation, scalar-generated vector and tensor contributions may cause systematic errors in the higher-order calculations.

V. GAUGE ISSUES IN THE SOLUTION

Gauge issue is a flaw in theory, and it has to be removed before any unique prediction in theory can be made in comparison to physical quantities. As we discussed in Sec. III, there exist remaining gauge modes in gauge choice II. We show that despite their presence, the matter density fluctuation in gauge choice II can be made independent to the second order in perturbation.

A. Linear-order gauge-transformation

Both gauge choices I and II in Table I take as the temporal gauge condition the comoving gauge $\mathcal{T}_\alpha^0 = 0$, which to the linear order in perturbation imposes the vector component of

² Even in the Newtonian dynamics, it is difficult to obtain an exact analytic solution for the matter density fluctuation, unless the Universe is a Einstein-de Sitter (EdS) universe with maximum symmetry. However, it is well-known that good approximate solutions are available in analytic form, when the growth factor D and the logarithmic growth rate f are used in the perturbative approach.

the four velocity vanishes $\tilde{v}_\alpha = v_\alpha = 0$ and the scalar component satisfies

$$\tilde{v}_{,\alpha} = v_{,\alpha} + T_{,\alpha} = 0. \quad (67)$$

Furthermore, the momentum constraint of the pressureless fluid indicates that the flow follows the geodesic in Eq. (24)

$$0 = N_{,\alpha} = \mathcal{A}_{,\alpha}, \quad (68)$$

where we used the ADM relation to the metric perturbations in Appendix A. The temporal gauge freedom is indeed constrained by the comoving gauge condition, only to be a scale-independent, but time-dependent function $T = T(t)$, which can be used to set $\mathcal{A} = 0$ by specifying $T = c/a(t)$, where c is some (unspecified) constant. This constant is further removed ($c = 0$, hence $T = 0$) by the conservation of the curvature perturbation $\dot{\varphi} = 0$.

Following these series of gauge transformations, both gauge choices have vanishing metric perturbation $\mathcal{A} = 0$ in time coordinates, and the time lapse of the ADM variable is $N = 1$. However, this is a deliberate gauge choice, not *automatically* imposed by the comoving gauge condition. Two gauge choices in Table I differ in the spatial gauge condition, but the remaining spatial gauge mode in gauge choice II has no impact on the matter density fluctuation δ_m and the expansion perturbation κ as shown in Eq. (33). This is further borne out by the equivalence of the nonlinear evolution equations (51) at the linear order.

B. Second-order gauge-transformation

To the second order in perturbation, the temporal gauge freedom can be completely removed ($T = 0$) for both gauge choices in Table I in a similar way to the linear-order case by setting $N = 1$ from the geodesic condition, in addition to the comoving gauge condition $v = v_\alpha = 0$. This implies that the metric perturbation in the time component vanishes $\mathcal{A} = 0$ for gauge choice II, while only the combination vanishes for gauge choice I

$$\mathcal{A} + \frac{1}{2}\mathcal{B}^\alpha \mathcal{B}_\alpha = 0, \quad (69)$$

where we used $\mathcal{A}^{(1)} = 0$ (see Appendix A). However, the above combination can be set to be a non-vanishing, but scale-independent function.

Regarding the spatial gauge condition, gauge choice I removes $L^{(1)} = L_\alpha^{(1)} = 0$ in Eq. (33), and the spatial metric perturbation in this case transforms to the second order in perturbation as [16, 20]

$$\tilde{\mathcal{C}}_{\alpha\beta}^{(2)} = \mathcal{C}_{\alpha\beta}^{(2)} - \mathcal{L}_{(\alpha|\beta)}^{(2)}. \quad (70)$$

Therefore, the decomposed perturbations transform as

$$\tilde{\gamma}^{(2)} = \gamma^{(2)} - L^{(2)}, \quad \tilde{\mathcal{C}}_\alpha^{(2)} = \mathcal{C}_\alpha^{(2)} - L_\alpha^{(2)}, \quad (71)$$

and the spatial gauge freedom can be completely removed in gauge choice I by imposing $\gamma = C_\alpha = 0$ to the second order in perturbation.

By contrast, gauge choice II sets $L^{(1)'} = L_\alpha^{(1)'} = 0$ in Eq. (33), and the off-diagonal metric perturbation in this case transforms as

$$\tilde{\mathcal{B}}_\alpha^{(2)} = \mathcal{B}_\alpha^{(2)} + \mathcal{L}_\alpha^{(2)'} - \mathcal{B}_{\alpha|\beta} \mathcal{L}^\beta - \mathcal{B}_\beta \mathcal{L}^\beta|_\alpha. \quad (72)$$

Therefore, the spatial gauge freedom is constrained in gauge choice II as $\mathcal{L}_\alpha^{(2)'} = 0$, and the residual spatial gauge mode remains as

$$L = L^{(1,2)}(\mathbf{x}), \quad L_\alpha = L_\alpha^{(1,2)}(\mathbf{x}). \quad (73)$$

In gauge choice I, no gauge freedom remains, and the solutions of the nonlinear Eqs. (51) and hence Eqs. (57) are uniquely determined. In gauge choice II, the remaining spatial gauge freedom can affect the solution, as the matter density fluctuation and the expansion transform to the second order in perturbation as [19]

$$\begin{aligned} \tilde{\delta}_m^{\text{II}}(\mathbf{x}, t) &= \delta_m^{\text{II}}(\mathbf{x}, t) - \nabla \delta_m^{\text{II}} \cdot \mathcal{L}^\alpha(\mathbf{x}), \\ \tilde{\kappa}^{\text{II}}(\mathbf{x}, t) &= \kappa^{\text{II}}(\mathbf{x}, t) - \nabla \kappa^{\text{II}} \cdot \mathcal{L}^\alpha(\mathbf{x}). \end{aligned} \quad (74)$$

It is now apparent that the solutions δ_m and κ in gauge choice II are *not* uniquely determined due to the arbitrary scale-dependent function $\mathcal{L}_\alpha(\mathbf{x})$. However, it appears that the solutions in Eq. (66) are uniquely determined in gauge choice II, despite the presence of gauge modes. Since the time-independent gauge modes in Eq. (74) are multiplied by the linear-order solutions, it vanishes in the left-hand side of Eqs. (59) and (60). In other words, the solutions in Eq. (66) are obtained by projecting out the remaining gauge modes in Eq. (74).

VI. PHYSICAL INTERPRETATION OF THE SOLUTIONS

The matter density fluctuations $\delta_m^{\text{I,II}}$ in two gauge choices represent the matter density fluctuation $\delta_m^{t_p}$ in the same proper-time hypersurface, and they both lack any gauge issues. At first glance, this conclusion appears odd, because the proper-time hypersurface of non-relativistic matter is physically well-defined and unique, yet solutions in two gauge choices differ as shown in Eqs. (65) and (66). The power spectrum and the bispectrum of the matter density fluctuations in these gauge choices are computed in [40, 43], showing the clear difference in two gauge choices. As we discussed in Sec. IV, the relativistic dynamics in gauge choices I and II are identical to the Eulerian and the Lagrangian Newtonian dynamics, respectively. However, we again emphasize that these correspondences are valid in the absence of linear-order vector or tensor, and pure relativistic corrections appear beyond the second-order in perturbations.

At the linear order, the gauge choices in Table I are identical, and the matter density fluctuations are also equivalent $\delta_m^{\text{I}} = \delta_m^{\text{II}}$ (and $\kappa^{\text{I}} = \kappa^{\text{II}}$), which can be obtained from the Boltzmann codes such as CMBFAST [31], CAMB [44], and

CLASS [45]. The difference arises beginning at the second order in perturbation, and the critical difference can be found in the large-scale limit of their kernels:

$$\begin{aligned} \lim_{\mathbf{k} \rightarrow 0} F_2^{\text{I}}(\mathbf{q}, \mathbf{k} - \mathbf{q}) &= \frac{3 - 5\mu^2}{7} \frac{k^2}{q^2} + \mathcal{O}(k^3), \\ \lim_{\mathbf{k} \rightarrow 0} F_2^{\text{II}}(\mathbf{q}, \mathbf{k} - \mathbf{q}) &= 1 + \frac{2(\mu^2 - 1)}{7} \frac{k^2}{q^2} + \mathcal{O}(k^3), \end{aligned} \quad (75)$$

where $\mu = \mathbf{k} \cdot \mathbf{q} / kq$. The kernel for gauge choice I vanishes as k^2 in the large-scale limit, while the kernel for gauge choice II becomes unity.

It is argued [46] that any nonlinear correction to the initial density field has to scale as wavenumber with power no less than two in the large-scale limit, since gravity respects the mass and the momentum conservation. The matter density fluctuation δ_m^{II} in gauge choice II in this respect violates the mass conservation, which essentially is due to the absence of the dipole term in Eq. (66). This can be further elaborated by considering the ensemble average of the matter density fluctuations:

$$\langle \delta_m^{(2)}(t, \mathbf{x}) \rangle = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} P(t, \mathbf{q}) F_2(\mathbf{q}, -\mathbf{q}) = \begin{cases} 0 & \text{for I} \\ \sigma_m^2 & \text{for II} \end{cases}, \quad (76)$$

where $P(t, \mathbf{q})$ is the linear matter power spectrum and $\sigma_m^2 = \langle \delta_m^2 \rangle$ is the unsmoothed rms fluctuation.

To the second order, the matter density fluctuation in gauge choice I, therefore, properly represents the “mean” matter density and the fluctuation around the mean:

$$\rho_m(t, \mathbf{x}) = \bar{\rho}_m(t) (1 + \delta_m^{\text{I}}), \quad \langle \rho_m(t, \mathbf{x}) \rangle = \bar{\rho}_m(t), \quad (77)$$

and it is noted that the mean matter density $\bar{\rho}_m$ is based on the coordinate time due to symmetry and its equality to the ensemble average is *not* by the definition. By contrast, the matter density fluctuation in gauge choice II has non-vanishing mean

$$\rho_m(t, \mathbf{x}) = \bar{\rho}_m(t) (1 + \delta_m^{\text{II}}), \quad \langle \rho_m(t, \mathbf{x}) \rangle = \bar{\rho}_m(t) (1 + \sigma_m^2), \quad (78)$$

as is derived in Eq. (76). The local observer sitting at the flow of non-relativistic matter has no way to obtain the “mean” matter density $\langle \rho_m \rangle$ averaged over the proper-time hypersurface. However, the mean matter density $\bar{\rho}_m(\tau)$ in the homogeneous universe can be estimated by using the proper time τ of the observer. Therefore, the correct matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface is represented by the matter density fluctuation δ_m^{I} in gauge choice I.

However, it remains still puzzling that both gauge choices as shown in Sec. III represent the physically well-defined hypersurface of non-relativistic matter flow, yet the matter density fluctuations differ without any gauge issues present. The resolution can be found by considering a transformation from gauge choice II to gauge choice I. Both gauge choices share the common time coordinates $t^{\text{I}} = t^{\text{II}} (T = 0)$, but differ only in the spatial coordinates

$$\mathbf{x}^{\text{I}} = \mathbf{x}^{\text{II}} + \mathcal{L}^\alpha(t, \mathbf{x}). \quad (79)$$

According to Eq. (33), we derive the linear-order gauge transformation as

$$L = \gamma^{\text{II}}, \quad L' = \frac{1}{a} \chi, \quad L_\alpha = C_\alpha^{\text{II}}, \quad L'_\alpha = \Psi_\alpha, \quad (80)$$

where Ψ_α is gauge invariant and χ is spatially invariant to the linear order. Integrating over time, the spatial transformation is obtained as

$$\mathcal{L}^\alpha = \int^t \frac{dt}{a} \left(\frac{1}{a} \nabla \chi + \Psi^\alpha \right) + \nabla \gamma^{\text{II}}(\mathbf{x}) + C_\alpha^{\text{II}}(\mathbf{x}), \quad (81)$$

where the integral term represents the time-dependent physical modes. The remaining time-independent but scale-dependent metric perturbations represent the remaining gauge freedom in gauge choice II, and they do not affect the matter density fluctuation δ_m^{II} as shown in Sec. V (see [19]). As in Eq. (74), the matter density fluctuation and the expansion perturbation are related as

$$\begin{aligned} \delta_m^{\text{I}}(\mathbf{x}, t) &= \delta_m^{\text{II}}(\mathbf{x}, t) - \nabla \delta_m^{\text{II}} \cdot \int^t \frac{dt}{a} \left(\frac{1}{a} \nabla \chi + \Psi^\alpha \right), \\ \kappa^{\text{I}}(\mathbf{x}, t) &= \kappa^{\text{II}}(\mathbf{x}, t) - \nabla \kappa^{\text{II}} \cdot \int^t \frac{dt}{a} \left(\frac{1}{a} \nabla \chi + \Psi^\alpha \right). \end{aligned} \quad (82)$$

It is now apparent that the spatial transformation \mathcal{L}^α in Eq. (81) is nothing but the spatial drift δx^α of non-relativistic matter in Eq. (39). Therefore, the difference between the gauge choices is that while they both represent the same proper-time hypersurface, their spatial coordinates differ in a way that the re-labeling of the spatial coordinate in gauge choice II violates the mass conservation at the second order in perturbation.

Furthermore, the spatial coordinate transformation in Eq. (79) can be viewed as a transformation from the Lagrangian frame (gauge choice II) to the Eulerian frame (gauge choice I). As we discussed in Sec. III, the spatial drift δx^α of non-relativistic matter in gauge choice II vanishes, such that the coordinate \mathbf{x}^{II} in Eq. (79) is identical to the initial position \mathbf{q} at very early time and the spatial transformation vector $\mathcal{L}^\alpha(t, \mathbf{x})$ corresponds to the Lagrangian displacement vector Ψ :

$$\mathbf{x}(t, \mathbf{q}) = \mathbf{q} + \Psi(t, \mathbf{q}), \quad (83)$$

where Ψ should be distinguished from the vector perturbation Ψ_α . In particular, the nonlinear evolution equation (51) at the linear order yields

$$\dot{\delta} = \kappa = -\frac{\Delta}{a^2} \chi, \quad (84)$$

and the spatial transformation vector in Eq. (81) is then related to the linear-order matter density fluctuation as

$$\mathcal{L}^\alpha = -\Delta^{-1} \nabla \delta_m(t, \mathbf{x}) + c(\mathbf{x}) = \Psi(t, \mathbf{x}) + c(\mathbf{x}), \quad (85)$$

where $c(\mathbf{x})$ is a scale-dependent integration constant and we ignored the vector contribution to the transformation.

Compared to δ_m^I in Eq. (82), the matter density fluctuation in gauge choice II is further compensated by the displacement vector:

$$\begin{aligned} & [\nabla \delta_m^{II} \cdot \Psi](t, \mathbf{k}) \\ &= -D^2(t) \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left[\frac{\mathbf{q}_1 \cdot \mathbf{q}_2}{2q_1 q_2} \left(\frac{q_1}{q_2} + \frac{q_2}{q_1} \right) \right] \delta(\mathbf{q}) \delta(\mathbf{k} - \mathbf{q}), \end{aligned} \quad (86)$$

which eliminates the dipole term in Eq. (65), leading to the kernel F_2^{II} in Eq. (66).

With this understanding, it is evident that gauge choices I and II describe the same system of irrotational non-relativistic matter flows, but in different perspectives: Eulerian versus Lagrangian. Furthermore, it is noted that as shown in Eq. (82) the matter density fluctuation in gauge choice II is *not* the one in Lagrangian perturbation theory, in which the matter density fluctuation respects the mass and the momentum conservation.

VII. GALAXY BIAS IN GENERAL RELATIVITY

Having identified the correct temporal and spatial gauge choices for the proper-time hypersurface, we are in a good position to discuss galaxy bias in the context of general relativity. Galaxy bias refers to the relation between the galaxy number density and the underlying matter distribution, and this relation is physically well-defined.

Due to the complexity of galaxy formation physics on small scales, biasing schemes are naturally effective descriptions, valid on large scales, which is the scale of our primary interest. The linear bias model (e.g., [9]) is that the galaxy number density fluctuation δ_g^{int} is proportional to the matter density fluctuation δ_m with constant bias factor b on large scales, where the physical galaxy number density n_g is separated into the mean and the fluctuation around it in a given coordinate system as

$$n_g = \bar{n}_g(\tau)(1 + \delta_g^{\text{int}}). \quad (87)$$

In general relativity, since the matter density fluctuation δ_m is gauge-dependent, the linear bias model makes little sense as long as the hypersurface for δ_m remains unspecified. In the context of general relativity, the linear bias model was recasted [3, 5, 6, 10, 11] to be valid in the proper-time hypersurface, and in terms of our notation τ in Eq. (87) is literally the proper-time and the galaxy number density fluctuation is

$$\delta_g^{\text{int}} = b \delta_m^{I,II}, \quad (88)$$

where $\delta_m^I = \delta_m^{II}$ at the linear order.

The linear biasing model can be naturally extended to the second order in perturbation as

$$\delta_g^{\text{int}} = b \delta_m^I, \quad (89)$$

and using Eqs. (76) and (77) the galaxy number density fluctuation satisfies

$$(\delta_g^{\text{int}})_\tau = 0, \quad \bar{n}_g(\tau) = \langle n_g \rangle_\tau, \quad (90)$$

where the ensemble average in this case is written as the spatial average in the proper-time hypersurface. It should be noted from Eqs. (76) and (78) that the above relation for n_g is violated in gauge choice II.

Beyond the linear order in Newtonian dynamics, local biasing models (e.g., [47, 48]) are frequently used, in which the galaxy number density fluctuation is a nonlinear function of the matter density fluctuation to be expanded in a Taylor series. At higher order, however, additional non-local terms can be included in biasing [49], and it was shown (e.g., [50–53]) that consistent renormalization of galaxy bias requires the presence of non-local derivative terms in addition to the local terms. At the second order, the additional non-local term is the contraction $s^2 = s_{ij}s_{ij}$ of the gravitational tidal tensor [49], and its evidence was measured [54] in simulations, where

$$s_{ij} \equiv \nabla_i \nabla_j \phi - \frac{1}{3} \delta_{ij}^K \delta_m = \left[\nabla_i \nabla_j \Delta^{-1} - \frac{1}{3} \delta_{ij}^K \right] \delta_m, \quad (91)$$

the normalization is $\nabla^2 \phi = \delta_m$, and δ_{ij}^K is the Kronecker delta.

To the second order, these quadratic terms in galaxy bias can be readily implemented to the relativistic framework, because we only need to consider them at the linear order:

$$\delta_m \rightarrow \delta_m^{I,II}, \quad \phi \rightarrow \frac{-\varphi_\chi}{4\pi G \bar{\rho}_m a^2}, \quad (92)$$

where the curvature potential φ_χ in the conformal Newtonian gauge is related to the gauge choices in Table I as $\varphi_\chi = \varphi - H\chi$ at the linear order.³

Therefore, the intrinsic fluctuation δ_g^{int} of the galaxy number density in general relativity can be written to the second order in perturbation as

$$\delta_g^{\text{int}} = b_1 \delta_m^I + \frac{1}{2} b_2 [(\delta_m^I)^2 - \sigma_m^2] + b_{s^2} [s^2 - \langle s^2 \rangle], \quad (93)$$

where $\langle s^2 \rangle = 2\sigma_m^2/3$ and it is noted that $\delta_m^I = \delta_m^{I(1)} + \delta_m^{I(2)}$. Individual variables in Eq. (93) are gauge-invariant, and of course they can be computed in other choices of gauge conditions, in which calculations become more involved.

Given the full second-order treatment in this paper, we briefly touch on the third-order galaxy bias in general relativity. At the third order in perturbation, the additional non-local terms are the cubic combination of the matter density fluctuation δ_m^I , the gravitational tidal tensor s_{ij} , and the velocity tidal tensor t_{ij} , such as $(\delta_m^I)^3$, $s^2 \delta_m^I$, s^3 , and $s_{ij} t_{ij}$ [49], where the velocity tidal tensor is,

$$t_{ij} \equiv \left[\nabla_i \nabla_j \Delta^{-1} - \frac{1}{3} \delta_{ij}^K \right] (\theta_N - \delta_m), \quad (94)$$

³ In terms of metric perturbation, the conformal Newtonian gauge (also known as the Poisson gauge or the longitudinal gauge) is defined with the temporal gauge condition $\chi = 0$ and the spatial gauge condition $\gamma = 0$. The notation is written in a way that the gauge invariant variable $\varphi_\chi = \varphi - H\chi$ becomes the curvature perturbation φ in the conformal Newtonian gauge ($\chi = 0$). The scalar shear component of the normal observer is $\sigma_{\alpha\beta} = \chi_{,\alpha|\beta} - \bar{g}_{\alpha\beta} \Delta\chi/3$, and hence the temporal gauge condition $\chi = 0$ is often called the zero-shear gauge.

non-vanishing only at the second order and the normalization in Newtonian dynamics is $\nabla^2 \theta_N = \delta_m$. Noting that the perturbation variables differ at the second order, we can readily identify the normalized divergence as

$$\theta_N \rightarrow -\frac{1}{\mathcal{H}f} \Theta^I = \frac{\kappa^I}{Hf}, \quad (95)$$

and of course $\delta_m = \delta_m^I$ in the relativistic framework. Another non-local term in galaxy bias that is by itself at the third order in perturbation is the scalar deviation [49]:

$$\psi = \theta_N - \delta_m - \frac{2}{7}s^2 + \frac{4}{21}\delta_m^2, \quad (96)$$

which vanishes up to the second-order in perturbation. We speculate that the scalar deviation term ψ may be identified by using the expansion perturbation κ^I as

$$\psi \rightarrow \frac{\kappa^I}{Hf} - \delta_m^I - \frac{2}{7}s^2 + \frac{4}{21}(\delta_m^I)^2, \quad (97)$$

at the third order.

VIII. DISCUSSION

We have computed the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface of non-relativistic matter flows to the second order in perturbation. It is identical to the matter density fluctuation in the temporal comoving gauge and the spatial C-gauge (gauge choice I in Table I). The commonly used matter density fluctuation in the temporal comoving gauge and the spatial B-gauge ($N = 1$, $N^\alpha = 0$; gauge choice II) violates the mass conservation, while it is gauge-invariant and also represents that in the proper-time hypersurface. We have provided physical understanding of each gauge condition by deriving the geodesic path of the comoving observer, solving the nonlinear evolution equations, and providing connections between gauge conditions. Drawing on this finding, we have provided the second-order galaxy biasing in general relativity, incorporating the nonlinear local and non-local terms that should be present for consistent renormalization of galaxy bias. The second-order galaxy biasing in this work provides an essential ingredient of the second-order relativistic description of galaxy clustering [16].

Non-relativistic matter responds only to gravity, following geodesic path and building up nonlinearity over time. When the density fluctuation becomes enormous $\delta_m \geq 200$, the gravitationally bound objects form, and the trajectories of non-relativistic matter are entangled at the same time. However, apart from these highly nonlinear regions and caustics, the flows of non-relativistic matter are non-intersecting and well-defined, in particular on large scales, but well into quasi-linear scales, which is the main reason the Zel'dovich approximation [55] or its variants are highly successful in describing nonlinearity. Therefore, on large scales, which is the scale of primary interest of this work, the proper-time hypersurface of non-relativistic matter flows is physically well-defined,

and the matter density fluctuation $\delta_m^{t_p}$ in the hypersurface can be computed without any ambiguity. Following the geodesic path of non-relativistic matter, we have derived the time drift $\delta\tau$ in Eq. (28) from the proper-time measured by the comoving observer of non-relativistic matter and used it to provide the formula in Eq. (30) for the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface.

Equation (30) can be evaluated with any choice of gauge condition, and it is often the case that the gauge choices in Table I are adopted to compute $\delta_m^{t_p}$ at the linear order using the popular Boltzmann codes such as CMBFAST [31], CAMB [44], and CLASS [45], in which the matter density fluctuations in those gauge conditions are equivalent. However, at the second order in perturbations, they are different, posing a critical question in galaxy bias — which matter density fluctuation represents the correct matter density fluctuation $\delta_m^{t_p}$ of the proper-time hypersurface that can be used in galaxy bias at the second order?

Gauge choice II is the commonly used *comoving-synchronous* gauge, in which there is no perturbation in the time coordinate $N = 1$ and the off-diagonal metric component $N_\alpha = 0$. With perturbations present only in the spatial metric, the coordinates follow the geodesic path, and the comoving observer is fixated at the spatial coordinates, such that the time drift of the comoving observer vanishes $\delta\tau = 0$ and the time coordinate in gauge choice II is synchronized with the proper time of non-relativistic matter to all orders in perturbation. Therefore, the matter density fluctuation δ_m^{II} is the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface. Despite the presence in this gauge choice, the remaining spatial gauge modes, which leave δ_m^{II} undetermined as in Eq. (74), can be projected out in δ_m^{II} by hand as in Eq. (66).

Gauge choice III is the original synchronous gauge. Despite the similarity to gauge choice II, additional temporal gauge mode remains in this gauge choice, even at the linear order, which needs to be removed by imposing the initial condition and thereby aligning it with gauge choice II. At the second order, gauge choice I has non-vanishing perturbation in the time component, and the coordinate observer is on non-inertial path. However, despite this shortcoming, the time coordinate of the comoving observer is still *synchronized* with the proper-time of non-relativistic matter in this gauge choice ($\delta\tau = 0$), and the matter density fluctuation δ_m^I is the matter density fluctuation $\delta_m^{t_p}$ in the proper-time hypersurface. Gauge freedom is completely fixed in this gauge choice, and δ_m^I in Eq. (65) is different from δ_m^{II} at the second order.

The physical resolution to the puzzle comes from the gauge transformation in each gauge condition. At the linear order, spatial gauge transformation is pure artifact due to the spatial symmetry in the background. However, at the second order, spatial gauge transformation is no longer a gauge artifact, but a physical transformation. Both gauge choices I and II describe the proper-time hypersurface of non-relativistic matter, but differ in spatial coordinates as illustrated in Eq. (79). In particular, the difference in the spatial coordinates is exactly the spatial drift of non-relativistic matter as in Eqs. (39) and (81) — gauge choice I describes the non-relativistic flows

in Eulerian frame, while gauge choice II in Lagrangian frame.

Precisely due to this difference in spatial displacement, the matter density fluctuations in both gauge choices differ, and one in gauge choice II violates the mass conservation, arising from the spatial distortion in coordinates. In the rest frame of non-relativistic matter, the proper-time is the only local observable that can be used to infer the mean matter density of the hypersurface. However, there exists non-vanishing large-scale mode present $\langle \delta_m^{\text{II}} \rangle = \sigma_m^2$ in gauge choice II, and hence δ_m^{II} cannot correctly describe the fluctuation around the mean $\langle \rho_m \rangle_t \neq \bar{\rho}_m(t)$ in the proper-time hypersurface. Similar conclusion was drawn in [38], in which the one-loop matter power spectrum in gauge choice II is computed.

With the proper identification of the matter density fluctuation in the proper-time hypersurface of non-relativistic matter flows, it becomes straightforward to generalize the nonlinear galaxy biasing schemes (e.g., [47–52, 54]) in Newtonian dynamics to those in the context of general relativity. In addition to the linear bias term, which requires the computation of second-order matter density fluctuation, additional nonlinear bias terms are quadratic at the second order, and hence their individual quantities need to be evaluated at the linear order. The additional local term δ_m^2 can be trivially implemented, and we have identified the additional nonlocal term s^2 from the gravitational tidal tensor as the Newtonian gauge curvature perturbation in Eqs. (91) and (92). The complete second-order galaxy biasing is given in Eq. (93). Additional third-order terms in galaxy bias are briefly discussed in Sec. VII.

Recently, the second-order relativistic description of galaxy clustering is computed by several groups [16–18]. In Bertacca et al. [17], they argue that the matter density fluctuation in the proper-time hypersurface is one δ_m^{II} in the comoving-time orthogonal gauge (gauge choice II in our terminology). As they correctly argue, the matter density fluctuation δ_m^{II} is gauge-invariant to the second order, if the remaining gauge modes are projected out. However, as we showed in this paper, δ_m^{II} does not properly represent the matter density fluctuation with the mean at the local proper time, violating the mass conservation. Yoo and Zaldarriaga [16] advocated the proper-time hypersurface for the second-order galaxy biasing scheme, and this current work completes the second-order relativistic description in [16] by providing the physical ground for galaxy bias.

Given the rapid development of current and future galaxy surveys and the particular emphasis on testing gravity on large scales, theoretical predictions need to be further improved by going beyond the linear theory, and subtle relativistic effects in galaxy clustering need to be fully utilized to take advantage of precision measurements of galaxy clustering. Equipped with the second-order galaxy biasing in this work, the second-order general relativistic description of galaxy clustering [16] provides such natural theoretical framework, in which further applications can build on such as the computation of the galaxy three-point statistics for investigating the sensitivity of the relativistic effect to the primordial non-Gaussianity and the modification of gravity on large scales.

Acknowledgments

We acknowledge useful discussions with Zvonimir Vlah, Tobias Baldauf, David Wands, and Toni Riotto. J. Y. is supported by the Swiss National Science Foundation.

Appendix A: Metric perturbations and their relation to the ADM variables

Thorough second-order calculations are presented in Noh and Hwang [20] (see also [56]). Here we summarize the useful relations between metric perturbations and the ADM variables that are used in the text.

Given the FRW metric perturbations in Eq. (4), the shift vector and the induced spatial metric in Eq. (8) are trivially matched as

$$N_\alpha = -a\mathcal{B}_\alpha, \quad h_{\alpha\beta} = a^2(\bar{g}_{\alpha\beta} + 2\mathcal{C}_{\alpha\beta}). \quad (\text{A1})$$

To the second order in perturbation, the remaining ADM variables are derived as

$$\begin{aligned} N &= 1 + \mathcal{A} - \frac{1}{2}\mathcal{A}^2 + \frac{1}{2}\mathcal{B}^\alpha\mathcal{B}_\alpha, \\ h^{\alpha\beta} &= \frac{1}{a^2}(\bar{g}^{\alpha\beta} - 2\mathcal{C}^{\alpha\beta} + 4\mathcal{C}_\gamma^\alpha\mathcal{C}^{\beta\gamma}), \\ N^\alpha &= h^{\alpha\beta}N_\beta = \frac{1}{a}(-\mathcal{B}^\alpha + 2\mathcal{B}^\beta\mathcal{C}_\beta^\alpha). \end{aligned} \quad (\text{A2})$$

In terms of metric perturbations, the normal observer in Eq. (13) is

$$\begin{aligned} n^0 &= 1 - \mathcal{A} + \frac{3}{2}\mathcal{A}^2 - \frac{1}{2}\mathcal{B}^\alpha\mathcal{B}_\alpha, \\ n^\alpha &= \frac{1}{a}(\mathcal{B}^\alpha - \mathcal{A}\mathcal{B}^\alpha - 2\mathcal{C}^{\alpha\beta}\mathcal{B}_\beta), \end{aligned} \quad (\text{A3})$$

and the four velocity in Eq. (15) is

$$\begin{aligned} \delta u^0 &= -\mathcal{A} + \frac{3}{2}\mathcal{A}^2 + \frac{1}{2}V^\alpha V_\alpha - V^\alpha\mathcal{B}_\alpha, \\ u_0 &= -\left(1 + \mathcal{A} - \frac{1}{2}\mathcal{A}^2 + \frac{1}{2}V^\alpha V_\alpha\right), \\ u_\alpha &= a(V_\alpha - \mathcal{B}_\alpha + 2\mathcal{C}_{\alpha\beta}V^\beta), \end{aligned} \quad (\text{A4})$$

where δu^0 is derived by the normalization condition ($u^a u_a = -1$).

The perturbation to the trace of the extrinsic curvature ten-

server and the traceless part of the extrinsic curvature are

$$\begin{aligned} \kappa = \delta K = & 3H\mathcal{A} - \frac{1}{a} \left(\mathcal{B}^\alpha_{|\alpha} + \mathcal{C}^{\alpha\prime}_{\alpha} \right) + \frac{\mathcal{A}}{a} \left(\mathcal{B}^\alpha_{|\alpha} + \mathcal{C}^{\alpha\prime}_{\alpha} \right) \\ & - \frac{3}{2}H \left(3\mathcal{A}^2 - \mathcal{B}^\alpha \mathcal{B}_\alpha \right) + \frac{1}{a} \mathcal{B}^\beta \left(2\mathcal{C}^\alpha_{\beta|\alpha} - \mathcal{C}^\alpha_{\alpha|\beta} \right) \\ & + \frac{2}{a} \mathcal{C}^{\alpha\beta} \left(\mathcal{B}_{\alpha|\beta} + \mathcal{C}'_{\alpha\beta} \right), \end{aligned} \quad (\text{A5})$$

$$\begin{aligned} \sigma_{\alpha\beta} = & a \left(\mathcal{B}_{(\alpha|\beta)} + \mathcal{C}'_{\alpha\beta} \right) (1 - \mathcal{A}) - a\mathcal{B}_\gamma \left(2\mathcal{C}^\gamma_{(\alpha|\beta)} - \mathcal{C}_{\alpha\beta}{}^{|\gamma} \right) \\ & - \frac{2}{3}a\mathcal{C}_{\alpha\beta} \left(\mathcal{B}^\gamma_{|\gamma} + \mathcal{C}^{\gamma\prime}_{\gamma} \right) - \frac{a}{3}\bar{g}_{\alpha\beta} \left[\left(\mathcal{B}^\gamma_{|\gamma} + \mathcal{C}^{\gamma\prime}_{\gamma} \right) (1 - \mathcal{A}) \right. \\ & \left. - \mathcal{B}^\gamma \left(2\mathcal{C}^\delta_{\gamma|\delta} - \mathcal{C}^\delta_{\delta|\gamma} \right) - 2\mathcal{C}^{\gamma\delta} \left(\mathcal{B}_{\gamma|\delta} + \mathcal{C}'_{\gamma\delta} \right) \right] \\ = & -\bar{K}_{\alpha\beta}, \end{aligned} \quad (\text{A6})$$

and we derive the nonlinear terms in Eq. (51)

$$\begin{aligned} \sigma_{ab}\sigma^{ab} = & \frac{1}{a^4} \left[\chi_{,\alpha|\beta} \chi^{\alpha|\beta} - \frac{1}{3} (\Delta\chi)^2 \right] + \frac{1}{a^2} \Psi_{\alpha|\beta} \Psi^{\alpha|\beta} \\ & + \dot{\mathcal{C}}_{\alpha\beta} \dot{\mathcal{C}}^{\alpha\beta} + \frac{2}{a^2} \chi_{,\alpha|\beta} \left(\frac{1}{a} \Psi^{\alpha|\beta} + \dot{\mathcal{C}}^{\alpha\beta} \right) \\ & + \frac{2}{a} \Psi_{\alpha|\beta} \dot{\mathcal{C}}^{\alpha\beta}, \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \frac{1}{3}\kappa^2 + \sigma_{ab}\sigma^{ab} = & \left(\frac{1}{a^2} \chi_{,\alpha|\beta} + \frac{1}{a} \Psi_{\alpha|\beta} + \dot{\mathcal{C}}_{\alpha\beta} \right) \\ & \times \left(\frac{1}{a^2} \chi^{\alpha|\beta} + \frac{1}{a} \Psi^{\alpha|\beta} + \dot{\mathcal{C}}^{\alpha\beta} \right). \end{aligned} \quad (\text{A8})$$

Appendix B: Gauge choice III — The synchronous gauge

Despite the similarity in the metric representation to gauge choice II, gauge choice III in Table I leaves gauge freedoms constrained only as in Eq. (45), and the metric perturbations other than $\mathcal{A} = \mathcal{B}_\alpha = 0$ are not uniquely determined by gauge choice III, even to the linear order in perturbation. Furthermore, since the comoving observer in this gauge choice differs from the normal observer, we cannot use the nonlinear equations (51) derived based on the covariant decomposition of the normal observer. It is noted that the energy momentum tensor in Eq. (21) is expressed in terms of the comoving ob-

server and the fluid quantities would be different if they are measured by the normal observer.

The Einstein equations in gauge choice III are

$$\begin{aligned} \kappa &= \frac{k^2}{a^2} \chi + 12\pi G \bar{\rho}_m a v, \\ \dot{\kappa} + 2H\kappa &= 4\pi G \rho_m \delta_m, \end{aligned} \quad (\text{B1})$$

and the conservation equations yield

$$\begin{aligned} \dot{\delta}_m &= \kappa - \frac{k^2}{a} v, \\ \dot{v} + H v &= 0. \end{aligned} \quad (\text{B2})$$

It becomes immediately clear that the differential equations in gauge choice III will become equivalent to those in Eq. (51) in gauge choice II, if the spatial scalar velocity vanishes $v = 0$, which decays in time according to the conservation equation. As discussed in Sec. IIID, this can be achieved by setting $v = 0$ at the initial condition and thereby effectively assuming gauge choice II. This is the gauge choice (and the initial condition) adopted in the Boltzmann codes such as CMBFAST [31], CAMB [44], and CLASS [45].

In gauge choice III, the remaining gauge modes affect the matter density fluctuation and the expansion perturbation as

$$\begin{aligned} \tilde{\delta}_m &= \delta_m + 3\mathcal{H}T = \delta_m + 3Hc_1(\mathbf{x}), \\ \tilde{\kappa} &= \kappa + \left(3\dot{H} + \frac{\Delta}{a^2} \right) aT = \kappa + 3\dot{H}c_1(\mathbf{x}) + \frac{\Delta}{a^2} c_1(\mathbf{x}), \end{aligned} \quad (\text{B3})$$

where $c_1(\mathbf{x})$ is an indeterminate scale-dependent function in Eq. (45). However, even in the presence of these gauge modes, the evolution equations (51) for gauge choice II can be used at the linear order, because the gauge modes happen to be proportional to the linear-order solutions H and \dot{H} in Eq. (51). However, this accidental coincidence is absent beyond the linear order, and one has to specifically adopt gauge choice II to proceed further.

The metric representation in Ma and Bertschinger [29] is related to our notation as

$$h_{ij} = 2\mathcal{C}_{ij}, \quad h = 6\varphi + 2\Delta\gamma, \quad \eta = -\varphi. \quad (\text{B4})$$

-
- [1] J. Yoo, A. L. Fitzpatrick, and M. Zaldarriaga, Phys. Rev. D **80**, 083514 (2009), arXiv:0907.0707.
 - [2] J. Yoo, Phys. Rev. D **82**, 083508 (2010), arXiv:1009.3021.
 - [3] A. Challinor and A. Lewis, Phys. Rev. D **84**, 043516 (2011), arXiv:1105.5292.
 - [4] C. Bonvin and R. Durrer, Phys. Rev. D **84**, 063505 (2011), arXiv:1105.5280.
 - [5] D. Jeong, F. Schmidt, and C. M. Hirata, Phys. Rev. D **85**, 023504 (2012), arXiv:1107.5427.
 - [6] J. Yoo, N. Hamaus, U. Seljak, and M. Zaldarriaga, Phys. Rev. D **86**, 063514 (2012), 1206.5809.
 - [7] J. Yoo, Phys. Rev. D **79**, 023517 (2009), arXiv:0808.3138.
 - [8] J. Yoo, Class. Quant. Grav. (2014), arXiv:1409.3223.
 - [9] N. Kaiser, Astrophys. J. Lett. **284**, L9 (1984).
 - [10] M. Bruni, R. Crittenden, K. Koyama, R. Maartens, C. Pitrou, and D. Wands, Phys. Rev. D **85**, 041301 (2012), arXiv:1106.3999.
 - [11] T. Baldauf, U. Seljak, L. Senatore, and M. Zaldarriaga, J. Cosmol. Astropart. Phys. **10**, 31 (2011), arXiv:1106.5507.
 - [12] D. Baumann and D. Green, Phys. Rev. D **85**, 103520 (2012), 1109.0292.
 - [13] K. Dimopoulos, M. Karciuskas, D. H. Lyth, and Y. Rodríguez,

- J. Cosmol. Astropart. Phys. **5**, 013 (2009), 0809.1055.
- [14] J. Beltrán Jiménez and A. L. Maroto, Phys. Rev. D **80**, 063512 (2009), 0905.1245.
- [15] J. Maldacena, J. High Energy Phys. **5**, 13 (2003), astro-ph/0210603.
- [16] J. Yoo and M. Zaldarriaga, Phys. Rev. D **90**, 023513 (2014), 1406.4140.
- [17] D. Bertacca, R. Maartens, and C. Clarkson, ArXiv e-prints (2014), 1406.0319.
- [18] E. Di Dio, R. Durrer, G. Marozzi, and F. Montanari, ArXiv e-prints (2014), 1407.0376.
- [19] J.-C. Hwang and H. Noh, Phys. Rev. D **73**, 044021 (2006), arXiv:0601041.
- [20] H. Noh and J.-C. Hwang, Phys. Rev. D **69**, 104011 (2004), arXiv:0305123.
- [21] J.-C. Hwang and H. Noh, Phys. Rev. D **76**, 103527 (2007), 0704.1927.
- [22] C. Uggla and J. Wainwright, Phys. Rev. D **90**, 043511 (2014), 1402.2464.
- [23] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
- [24] J. M. Bardeen, in *Cosmology and Particle Physics*, edited by L. Fang and A. Zee (Gordon and Breach, London, 1988), p. 1.
- [25] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, vol. 40 (Wiley, New York, 1962), arXiv:0405109.
- [26] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (W.H. Freeman and Co., San Francisco, ISBN 0-7167-0344-0, 1973).
- [27] J. Ehlers, *Proceedings of the mathematical-natural science of the Mainz academy of science and literature, translated in Gen. Rel. Grav.* **25**, 1225, 1993, vol. 792 (1961).
- [28] G. F. R. Ellis, in *General Relativity and Cosmology*, edited by R. K. Sachs (1971), pp. 104–182.
- [29] C.-P. Ma and E. Bertschinger, Astrophys. J. **455**, 7 (1995), arXiv:astro-ph/9401007.
- [30] M. Kasai, Physical Review Letters **69**, 2330 (1992).
- [31] U. Seljak and M. Zaldarriaga, Astrophys. J. **469**, 437 (1996), astro-ph/9603033.
- [32] J.-C. Hwang and H. Noh, Phys. Rev. D **72**, 044012 (2005), gr-qc/0412129.
- [33] J.-c. Hwang and H. Noh, J. Cosmol. Astropart. Phys. **12**, 003 (2007), 0704.2086.
- [34] A. Raychaudhuri, Phys. Rev. **98**, 1123 (1955).
- [35] J.-C. Hwang and H. Noh, Phys. Rev. D **59**, 067302 (1999), arXiv:9812007.
- [36] J.-C. Hwang and H. Noh, Phys. Rev. D **72**, 044011 (2005), arXiv/0412128.
- [37] F. Bernardeau, S. Colombi, E. Gaztañaga, and R. Scoccimarro, Phys. Rep. **367**, 1 (2002), arXiv:astro-ph/0112551.
- [38] J.-c. Hwang, H. Noh, D. Jeong, J.-O. Gong, and S. G. Biern, ArXiv e-prints (2014), 1408.4656.
- [39] M. Bruni, J. C. Hidalgo, N. Meures, and D. Wands, Astrophys. J. **785**, 2 (2014), 1307.1478.
- [40] S. G. Biern, J.-O. Gong, and D. Jeong, Phys. Rev. D **89**, 103523 (2014), 1403.0438.
- [41] C. Rampf, Phys. Rev. D **89**, 063509 (2014), 1307.1725.
- [42] M. H. Goroff, B. Grinstein, S. Rey, and M. B. Wise, Astrophys. J. **311**, 6 (1986).
- [43] D. Jeong, J.-O. Gong, H. Noh, and J.-c. Hwang, Astrophys. J. **727**, 22 (2011), 1010.3489.
- [44] A. Lewis, A. Challinor, and A. Lasenby, Astrophys. J. **538**, 473 (2000), arXiv:astro-ph/9911177.
- [45] J. Lesgourgues, ArXiv e-prints (2011), 1104.2932.
- [46] P. J. E. Peebles, *The large-scale structure of the universe* (Princeton University Press, Princeton, 1980).
- [47] A. S. Szalay, Astrophys. J. **333**, 21 (1988).
- [48] J. N. Fry and E. Gaztanaga, Astrophys. J. **413**, 447 (1993), astro-ph/9302009.
- [49] P. McDonald and A. Roy, J. Cosmol. Astropart. Phys. **8**, 20 (2009), 0902.0991.
- [50] P. McDonald, Phys. Rev. D **74**, 103512 (2006), arXiv:astro-ph/0609413.
- [51] F. Schmidt, D. Jeong, and V. Desjacques, Phys. Rev. D **88**, 023515 (2013), 1212.0868.
- [52] V. Assassi, D. Baumann, D. Green, and M. Zaldarriaga, ArXiv e-prints (2014), 1402.5916.
- [53] A. Kehagias, J. Noreña, H. Perrier, and A. Riotto, Nuclear Physics B **883**, 83 (2014), 1311.0786.
- [54] T. Baldauf, U. Seljak, V. Desjacques, and P. McDonald, Phys. Rev. D **86**, 083540 (2012), 1201.4827.
- [55] Y. B. Zel'dovich, Astron. Astrophys. **5**, 84 (1970).
- [56] K. A. Malik and D. Wands, Phys. Rep. **475**, 1 (2009), 0809.4944.